05 Functions I

Calculator Free

1. [4 marks: 2, 2]

Determine analytically if each of the following functions are one-to-one or many-to-one functions.

(a) $f(x) = \dfrac{1}{x^2}$

(b) $f(x) = \ln(1 + x)$

---

2. [6 marks: 3, 3]

Find the largest possible domain for each of the following functions to be one-to-one functions. In each case, state the corresponding range.

(a) $f(x) = x(x - 1)$

(b) $f(x) = -1 + \dfrac{1}{2}\sqrt{36 - 9(x-1)^2}$

Calculator Free

3. [5 marks: 3, 2]

Given that $f(x) = x^2 - 2$ and $g(x) = \sqrt{x+2}$.

(a) Find the rule for $g\,f(x)$.

(b) State the natural domain and range for $gf(x)$.

4. [5 marks: 1, 2, 2]

Given that $f(x) = \dfrac{1}{x+1}$ and $g(x) = x - 4$.

(a) State the natural domain for g.

(b) Explain clearly why the domain for g has to be restricted if the $f\,g$ is to be a function.

(c) State the largest possible domain for fg and the corresponding range.

Calculator Free

5. [4 marks] [TISC]

Let $f(x) = \sqrt{1-x^2}$ and $g(x) = \dfrac{1}{x-1}$.

Find the rule for the composite function $g f(x)$. State the domain for $g f(x)$.

6. [5 marks]

Let $f(x) = \sqrt{9-x}$ and $g(x) = 3^{x+1}$. Find the domain and range for $f(g(x))$.

Calculator Free

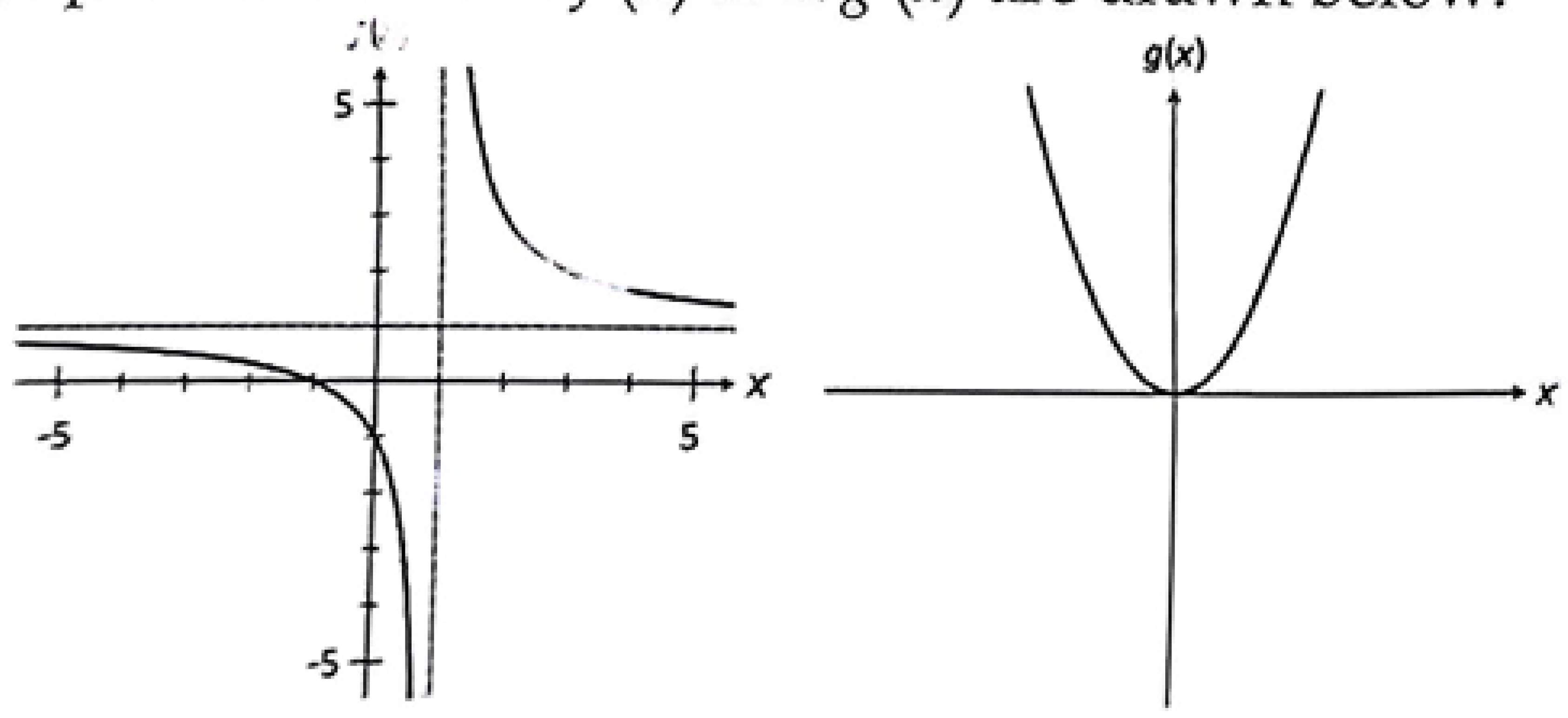
7. [6 marks: 4, 2] [TISC]

(a) Given that $f \circ g\,(x) = \dfrac{x^2+2}{x^2+1}$, and $f(x) = 1 + x^2$, find $g\,(x)$.

(b) Given that $g \circ f\,(x) = 3\,e^{2x+1}$, and $f(x) = 2x$, find $g\,(x)$.

8. [4 marks: 2, 2] [TISC]

The graphs of functions $f(x)$ and $g\,(x)$ are drawn below.



(a) Find the asymptote(s) of $g\,f\,(x)$.

(b) Find the range for $g\,f\,(x)$.

Calculator Free

9. [7 marks: 2, 2, 3] [TISC]

(a) Given that $f \circ f(x) = x + 4$, find $f(x)$.

(b) Given that $g \circ g(x) = x^4$, find $g(x)$.

(c) Explain clearly why it is not possible to find a real valued function $h(x)$ such that $h \circ h(x) = -x$.
[Hint: Let $h(x) = ax + b$ where $a, b \in \mathbb{R}$.]

10. [3 marks]

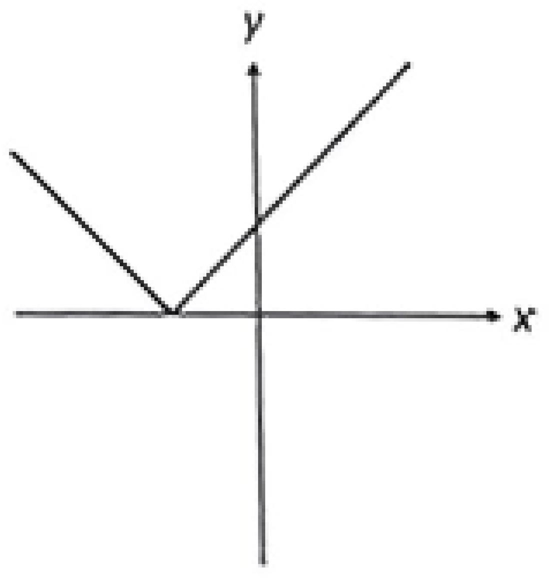
Let $f(x) = x^2$ and $g(x) = 1 + \sqrt{x}$.
Solve $f \circ g(x) = g \circ f(x)$.

Calculator Free

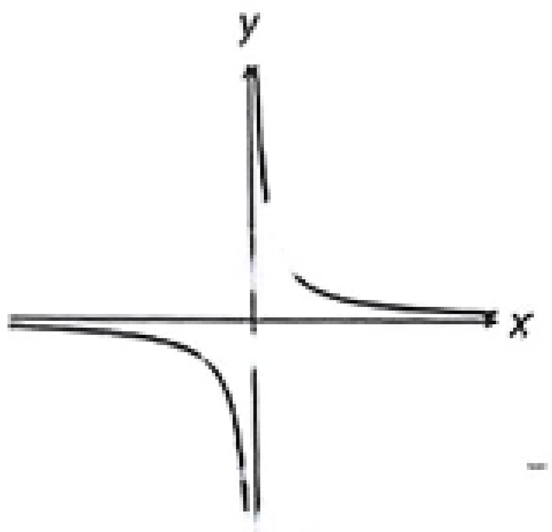
11. [6 marks: 2, 2, 2]

Determine with reasons if each of the following functions as drawn have inverse functions.

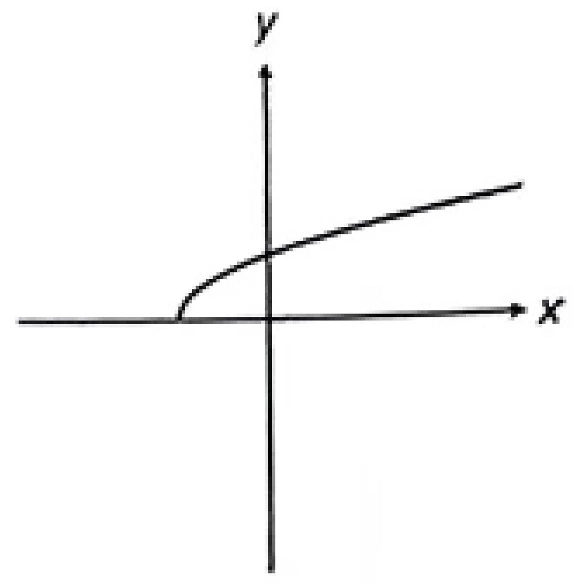
(a)



(b)



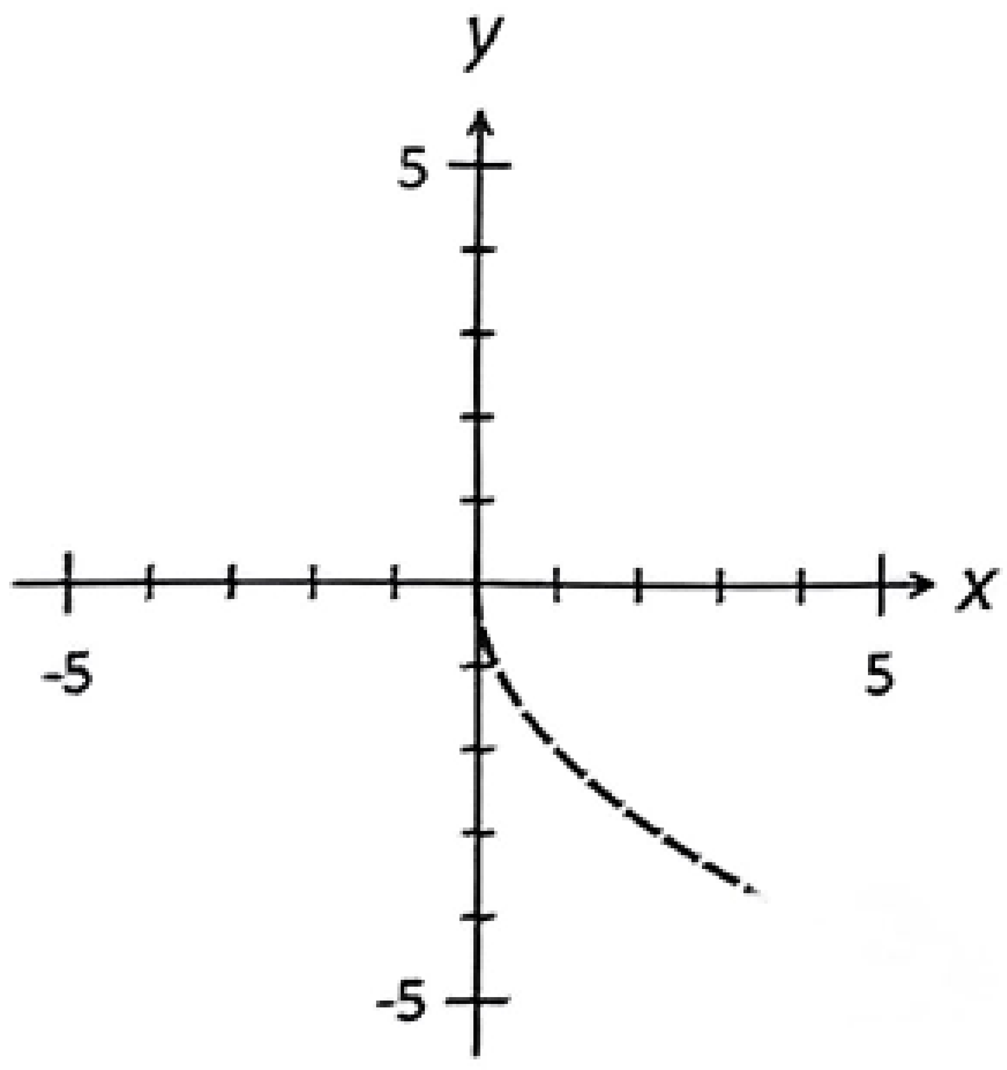
(c)



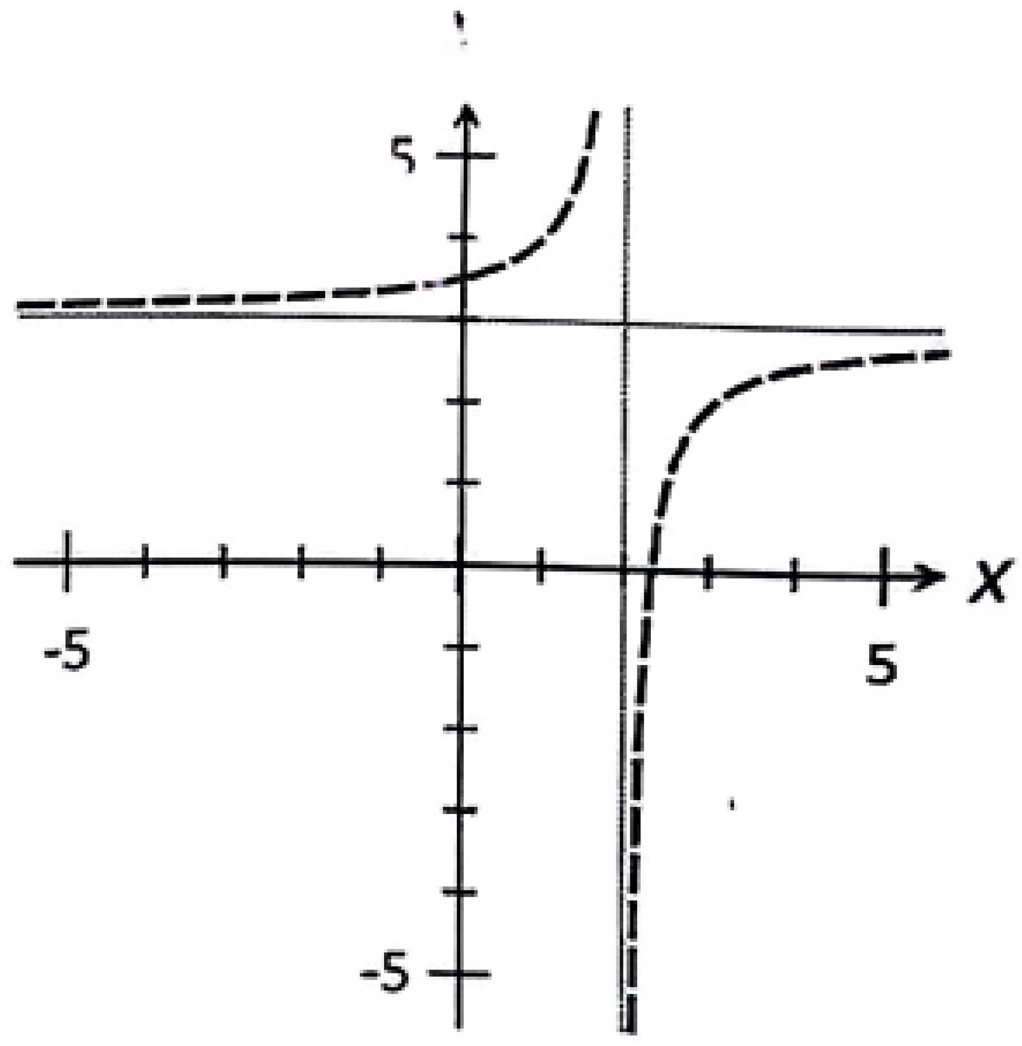
12. [6 marks: 2, 4]

The graph of $y = f(x)$ is shown in the accompanying diagrams. In each case, sketch the graph for $f^{-1}(x)$.

(a)



(b)



Calculator Free

13. [5 marks]

Prove that for the one-to-one functions f and g, $(f\,g)^{-1}(x) = g^{-1}f^{-1}(x)$.
[Hint: $h(h^{-1}(x)) = x$]

14. [8 marks: 2, 4, 2]

Consider the function $f(x) = \dfrac{1-x}{2+x}$.

(a) State the natural domain and range for $f(x)$.

(b) Find the rule for $f^{-1}(x)$.

(c) State the domain and range for $f^{-1}(x)$.

Calculator Free

15. [10 marks: 2, 2, 1, 3, 2]

Given that $f(x) = (x-1)^2$ where x is a real number.

(a) Find $f(0)$ and $f(2)$.

(b) Show clearly how you would use your answers in (a) to show that $f(x)$ does not have an inverse function.

(c) Find the largest possible domain for $f(x)$, consisting only of positive numbers, so that $f(x)$ has an inverse function.

(d) For the domain in (c), find the rule for the inverse for $f(x)$.

(e) For the domain in (c), find the domain and range for the inverse for $f(x)$.

Calculator Free

16. [6 marks: 1, 3, 2] [TISC]

Consider $f(x) = \sqrt{(x-1)^2}$.

(a) Find the largest possible domain for f so that f has an inverse function.

(b) Using your answer in (a) find the rule for the inverse of f.

(c) For your answer in (b), state the domain and range for the inverse of f.

---

17. [8 marks: 4, 4]

Let $f(x) = ln\,(1-x)$ and $g(x) = \dfrac{1}{x}$.

(a) Find the largest possible domain for f so that $g \circ f$ is a function.
State the accompanying range.

(b) Find x such that $g \circ f(x) = g^{-1} \circ f(x)$. Justify your answer.

Calculator Free

18. [6 marks: 2, 1, 3]

Consider $f(x) = e^{(x-1)^2}$.

(a) Show that the inverse of f is not a function.

(b) Find the largest possible domain consisting of both positive and negative numbers so that the inverse of f is a function.

(c) For the domain you specified in part (b), find the rule for the inverse of f.

---

19. [5 marks: 2, 3]

Consider $f(x) = \left| \dfrac{2x-1}{x-3} \right|$.

(a) Explain clearly why within its natural domain f does not have an inverse function.

Calculator Free

19. (b) For $f(x) = \left| \dfrac{2x-1}{x-3} \right|$ where $\frac{1}{2} \le x < 3$, find the rule for f^{-1}.

---

20. [6 marks: 3, 3]

Consider $f(x) = \sin 2x$ and $g(x) = \cos \dfrac{x}{2}$.

(a) $f(x)$ is a one-to-one function within the domain $-a \le x \le a$.
Determine the largest possible value for $|a|$.
Hence, determine the rule for $f^{-1}(x)$ and state the corresponding range.

(b) $g(x)$ is a one-to-one function within the domain $0 \le x \le b$.
Determine the largest possible value for b.
Hence, determine the rule for $g^{-1}(x)$ and state the corresponding range.

05 Functions I

Calculator Free

1. [4 marks: 2, 2]

Determine analytically if each of the following functions are one-to-one or many-to-one functions.

(a) $f(x) = \frac{1}{x^2}$

$f(-1) = f(1) = 1$. ✓
Hence, $f(x)$ is a many-to-one function. ✓

(b) $f(x) = ln(1 + x)$

For $f(a) = f(b)$, $ln(1 + a) = ln(1 + b)$
$\Rightarrow a = b$ ✓
Hence, $f(x)$ is a one-to-one function. ✓

2. [6 marks: 3, 3]

Find the largest possible domain for each of the following functions to be one-to-one functions. In each case, state the corresponding range.

(a) $f(x) = x(x - 1)$

$f(x)$ is symmetrical about $x = \frac{1}{2}$. ✓

Hence, largest possible domain is $[\frac{1}{2}, \infty)$ with corresponding range $[-\frac{1}{4}, \infty)$. ✓✓

OR $(-\infty, \frac{1}{2}]$ with range $[-\frac{1}{4}, \infty)$.

(b) $f(x) = -1 + \frac{1}{2}\sqrt{36 - 9(x-1)^2}$

$f(x)$ is symmetrical about $x = 1$. ✓
Hence, largest possible domain is $[1, 3]$ with corresponding range $[-1, 2]$. ✓✓
OR $[-2, 1]$ with range $[-1, 2]$.

Calculator Free

3. [5 marks: 3, 2]

Given that $f(x) = x^2 - 2$ and $g(x) = \sqrt{x+2}$.

(a) Find the rule for $g\,f(x)$.

$g\,f(x) = g(x^2 - 2)$
$= \sqrt{x^2 - 2 + 2}$ ✓
$= \sqrt{x^2}$ ✓
$= |x|$ ✓

(b) State the natural domain and range for $gf(x)$.

Domain $\mathbb{R}$. ✓
Range $[0, \infty)$ ✓

4. [5 marks: 1, 2, 2]

Given that $f(x) = \frac{1}{x+1}$ and $g(x) = x - 4$.

(a) State the natural domain for g.

Domain $\mathbb{R}$ ✓

(b) Explain clearly why the domain for g has to be restricted if the fg is to be a function.

$g(3) = -1$
$fg(3)$ is undefined. ✓
Hence, domain for fg has to be restricted to $\{x: x \neq 3,\ x \in \mathbb{R}\}$ ✓

(c) State the largest possible domain for fg and the corresponding range.

Domain $\{x: x \neq 3,\ x \in \mathbb{R}\}$ ✓
Range $\{y: y \neq 0,\ y \in \mathbb{R}\}$ ✓

Calculator Free

5. [4 marks] [TISC]

Let $f(x) = \sqrt{1-x^2}$ and $g(x) = \dfrac{1}{x-1}$.

Find the rule for the composite function $g\,f(x)$. State the domain for $g\,f(x)$.

$g\,f(x) = g[f(x)]$

$= \dfrac{1}{f(x)-1} = \dfrac{1}{\sqrt{1-x^2}-1}$ ✓

Clearly $\sqrt{1-x^2} \neq 1$

$x \neq 0$ ✓

Also, $1-x^2 \geq 0$

$\Rightarrow -1 \leq x \leq 1$ ✓

Hence, domain for $g\,f(x)$ is:

$\{x: -1 \leq x \leq 1$ where $x \neq 0,\ x \in \mathbb{R}\}$ ✓

6. [5 marks]

Let $f(x) = \sqrt{9-x}$ and $g(x) = 3^{x+1}$. Find the domain and range for $f(g(x))$.

$f(g(x)) = f(3^{x+1})$

$= \sqrt{9-3^{x+1}}$ ✓✓

Domain: $9 - 3^{x+1} \geq 0$

$3^2 \geq 3^{x+1}$

$2 \geq x+1$

Hence, domain is $\{x : x \leq 1,\ x \in \mathbb{R}\}$ ✓

Range: $\{y: 0 \leq y < 3,\ x \in \mathbb{R}\}$ ✓✓

Calculator Free

7. [6 marks: 4, 2] [TISC]

(a) Given that $f \circ g(x) = \dfrac{x^2+2}{x^2+1}$, and $f(x) = 1 + x^2$, find $g(x)$.

$f \circ g(x) = f(g(x)) = 1 + [g(x)]^2$

Hence: $1 + [g(x)]^2 = \dfrac{x^2+2}{x^2+1}$ ✓

$[g(x)]^2 = \dfrac{x^2+2}{x^2+1} - 1$ ✓

$g(x) = \sqrt{\dfrac{1}{x^2+1}}$ or $-\sqrt{\dfrac{1}{x^2+1}}$ ✓✓

(b) Given that $g \circ f(x) = 3\,e^{2x+1}$, and $f(x) = 2x$, find $g(x)$.

Let $u = 2x$.

Hence, $g \circ f(x) = g(f(x))$

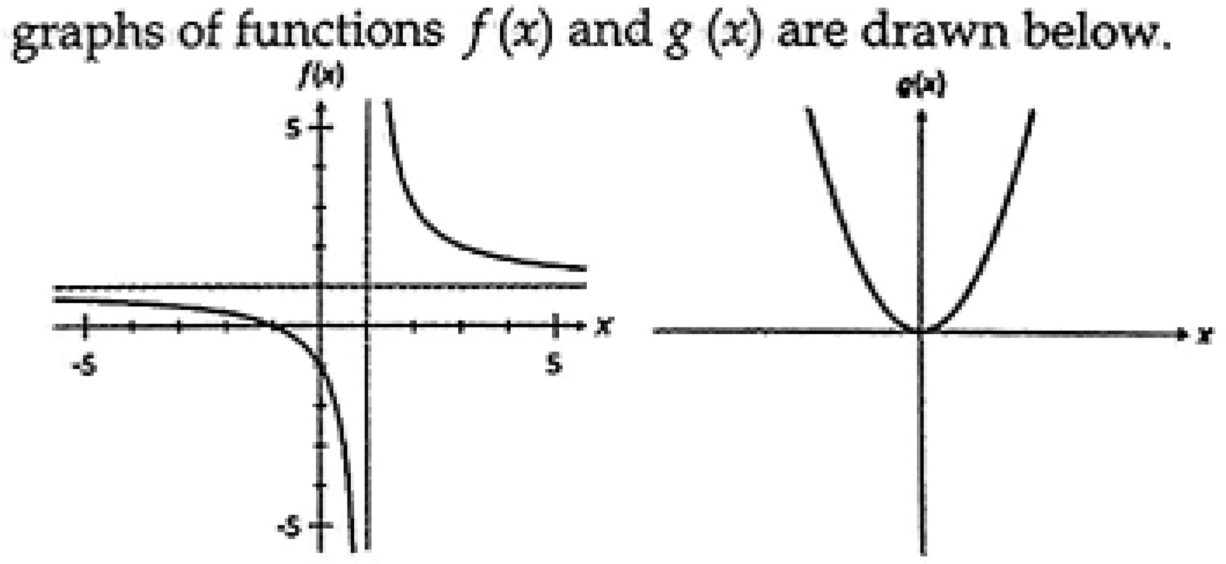
$= g(2x) = g(u)$ ✓

$\Rightarrow g(u) = 3\,e^{2x+1} = 3\,e^{u+1}$

Hence, $g(x) = 3\,e^{x+1}$ ✓

8. [4 marks: 2, 2] [TISC]

The graphs of functions $f(x)$ and $g(x)$ are drawn below.



(a) Find the asymptote(s) of $g\,f(x)$.

$x = 1$ and $y = 1$ ✓✓

(b) Find the range for $g\,f(x)$.

$\{y: y \geq 0, y \neq 1,\ y \in \mathbb{R}\}$ ✓✓

Calculator Free

9. [7 marks: 2, 2, 3] [TISC]

(a) Given that $f \circ f(x) = x + 4$, find $f(x)$.

By inspection: $f(x) = x + 2$ ✓✓

(b) Given that $g \circ g(x) = x^4$, find $g(x)$.

By inspection: $g(x) = x^2$ ✓✓

(c) Explain clearly why it is not possible to find a real valued function $h(x)$ such that $h \circ h(x) = -x$.
[Hint: Let $h(x) = ax + b$ where $a, b \in \mathbb{R}$.]

Let $h(x) = ax + b$ where $a, b \in \mathbb{R}$.
$\Rightarrow h(h(x)) = h(ax + b)$
$= a(ax + b) + b$
$= a^2x + (ab + b)$ ✓
But $h(h(x)) = -x$.
Compare x terms: $a^2 = -1$ ✓
$\Rightarrow a$ is not real. ✓
Hence, it is not possible.

---

10. [3 marks]

Let $f(x) = x^2$ and $g(x) = 1 + \sqrt{x}$.
Solve $f \circ g(x) = g \circ f(x)$.

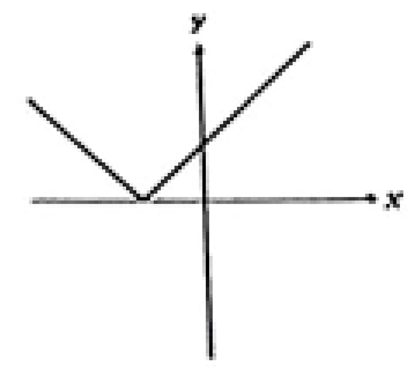
$f(g(x)) = g(f(x))$
$(1 + \sqrt{x})^2 = 1 + \sqrt{x^2}$ ✓✓
By inspection, $x = 0$ ✓

Calculator Free

11. [6 marks: 2, 2, 2]

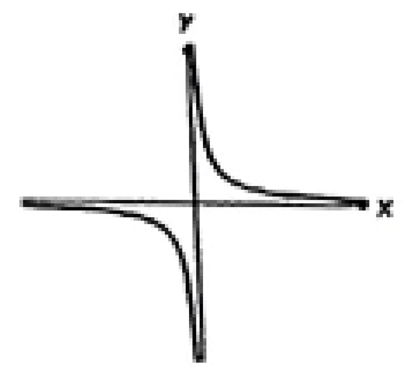
Determine with reasons if each of the following functions as drawn have inverse functions.

(a)



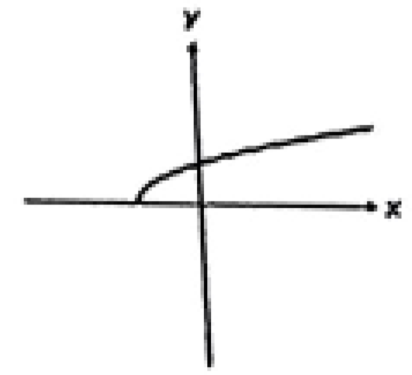
No! Fails the horizontal line test. ✓✓
(Not one to one)

(b)



Yes! Passes the horizontal line test. ✓✓
(one to one)

(c)



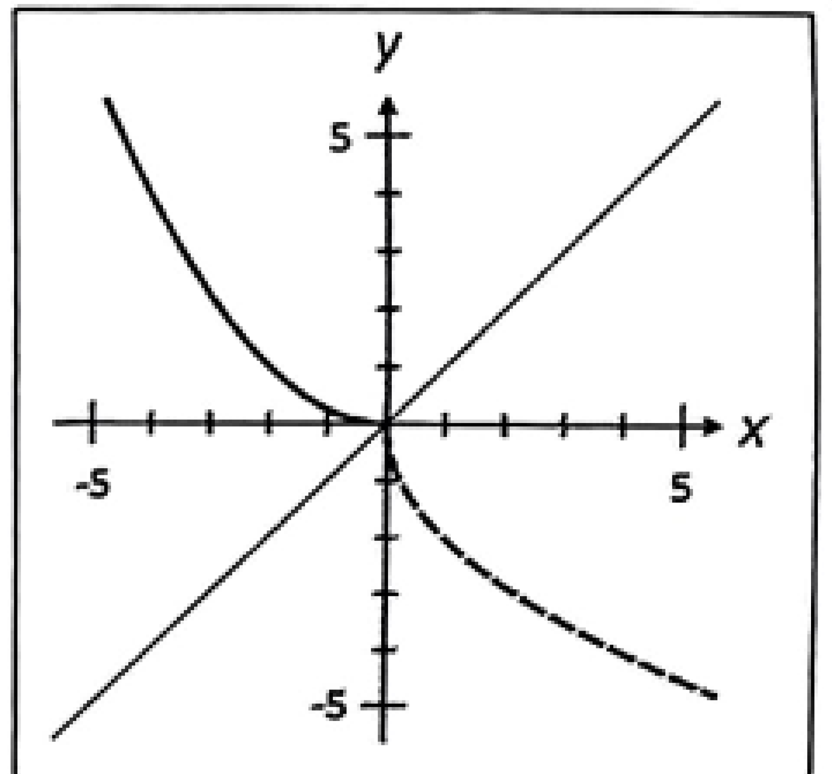
Yes! Passes the horizontal line test. ✓✓
(one to one)

---

12. [6 marks: 2, 4]

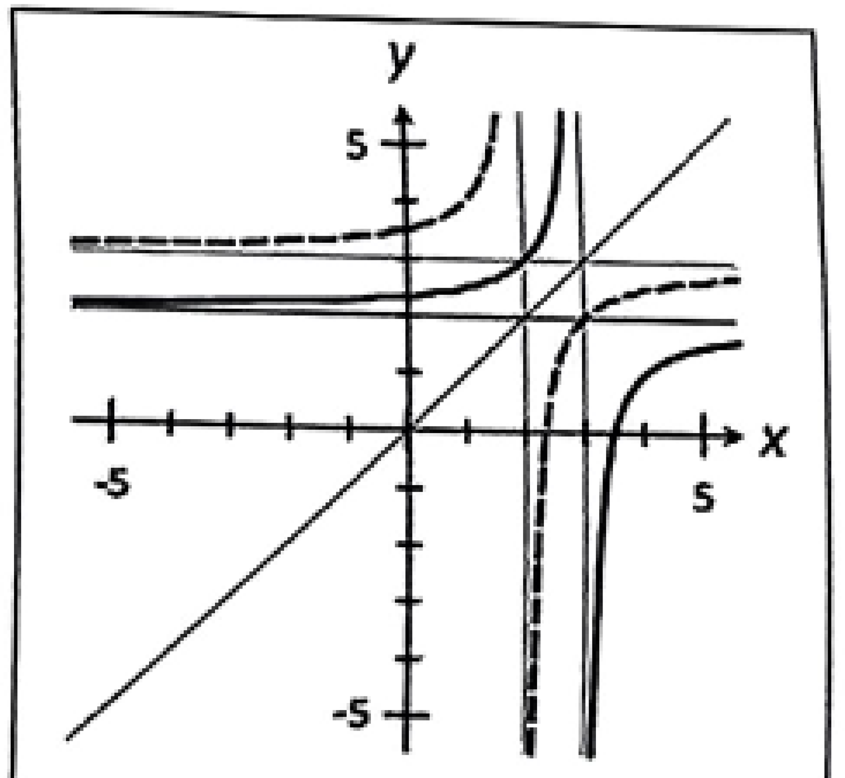
The graph of $y = f(x)$ is shown in the accompanying diagrams. In each case, sketch the graph for $f^{-1}(x)$.

(a)



Symmetrical about $y = x$ ✓
All correct ✓

(b)



Symmetrical about $y = x$ ✓
Asymptotes $y = 2$ & $x = 3$ ✓✓
All correct ✓

Calculator Free

13. [5 marks]

Prove that for the one-to-one functions f and g, $(f\,g)^{-1}(x) = g^{-1}f^{-1}(x)$.

[Hint: $h(h^{-1}(x)) = x$]

$f\,g((f\,g)^{-1})(x) = x$ ✓

$f^{-1}(f\,g((f\,g)^{-1}))(x) = f^{-1}(x)$ ✓

$g(f\,g)^{-1})(x) = f^{-1}(x)$ ✓

$g^{-1}(g(f\,g)^{-1}))(x) = g^{-1}(f^{-1}(x))$ ✓

$(f\,g)^{-1}(x) = g^{-1}f^{-1}(x)$ ✓

---

14. [8 marks: 2, 4, 2]

Consider the function $f(x) = \dfrac{1-x}{2+x}$.

(a) State the natural domain and range for $f(x)$.

Domain $\{x: x \neq -2,\ x \in \mathbb{R}\}$ ✓
Range $\{y: y \neq -1,\ y \in \mathbb{R}\}$ ✓

(b) Find the rule for $f^{-1}(x)$.

Let $y = \dfrac{1-x}{2+x}$.

$xy + x = 1 - 2y$ ✓

$x(1+y) = 1 - 2y$ ✓

$\Rightarrow \quad x = \dfrac{1-2y}{1+y}$. ✓

Hence, $f^{-1}(x) = \dfrac{1-2x}{1+x}$ ✓

(c) State the domain and range for $f^{-1}(x)$.

Domain $\{x: x \neq -1,\ x \in \mathbb{R}\}$ ✓
Range $\{y: y \neq -2,\ y \in \mathbb{R}\}$ ✓

Calculator Free

15. [10 marks: 2, 2, 1, 3, 2]

Given that $f(x) = (x-1)^2$ where x is a real number.

(a) Find $f(0)$ and $f(2)$.

$f(0) = 1$ ✓
$f(2) = 1$ ✓

(b) Show clearly how you would use your answers in (a) to show that $f(x)$ does not have an inverse function.

Since $f(0) = f(2) = 1$, $f(x)$ is a many to one function. ✓✓
Hence, $f(x)$ does not have an inverse function.

(c) Find the largest possible domain for $f(x)$, consisting only of positive numbers, so that $f(x)$ has an inverse function.

Domain $\{x: x \geq 1,\ x \in \mathbb{R}\}$ ✓

(d) For the domain in (c), find the rule for the inverse for $f(x)$.

Let $y = (x-1)^2$.

$x - 1 = \pm\sqrt{y}$

$x = 1 \pm \sqrt{y}$ ✓

Since $x \geq 1$, $x = 1 + \sqrt{y}$. ✓

Hence, $f^{-1}(x) = 1 + \sqrt{x}$ ✓

(e) For the domain in (c), find the domain and range for the inverse for $f(x)$.

Domain $\{x: x \geq 0,\ x \in \mathbb{R}\}$. ✓
Range $\{y: y \geq 1,\ y \in \mathbb{R}\}$. ✓

Calculator Free

16. [6 marks: 1, 3, 2] [TISC]

Consider $f(x) = \sqrt{(x-1)^2}$.

(a) Find the largest possible domain for f so that f has an inverse function.

$[1, \infty)$ or $(-\infty, 1]$ ✓

(b) Using your answer in (a) find the rule for the inverse of f.

For $x \geq 1$, $y = x - 1$ ✓✓
$\Rightarrow f^{-1}(x) = x + 1$ ✓

For $x \leq 1$, $y = -x + 1$ ✓✓
$\Rightarrow f^{-1}(x) = -x + 1$ ✓

(c) For your answer in (b), state the domain and range for the inverse of f.

For $x \geq 1$:
domain is $[0, \infty)$, ✓
range is $[1, \infty)$. ✓

For $x \leq 1$:
domain is $[0, \infty)$, ✓
range is $(-\infty, 1]$. ✓

---

17. [8 marks: 4, 4]

Let $f(x) = ln(1 - x)$ and $g(x) = \frac{1}{x}$.

(a) Find the largest possible domain for f so that $g \circ f$ is a function.
State the accompanying range.

$g(f(x)) = \frac{1}{f(x)} = \frac{1}{ln(1-x)}$ ✓
Hence, domain is $\{x: x < 1, x \neq 0, x \in \mathbb{R}\}$ ✓✓
Range is $\{y: y \neq 0, y \in \mathbb{R}\}$ ✓

(b) Find x such that $g \circ f(x) = g^{-1} \circ f(x)$. Justify your answer.

$g(x) = \frac{1}{x} \Rightarrow g^{-1}(x) = \frac{1}{x}$ ✓
$g^{-1}(x) = g(x)$ ✓
Hence, $g(f(x)) \equiv g^{-1}(f(x))$. ✓
Therefore, $g(f(x)) = g^{-1}(f(x))$
$\forall\ x < 1$ with $x \neq 0$. ✓

Mathematics Specialist Units 3 & 4 Revision Series *05 Functions I*

Calculator Free

18. [6 marks: 2, 1, 3]

Consider $f(x) = e^{(x-1)^2}$.

(a) Show that the inverse of f is not a function.

$f(0) = f(2) = e^1$ ✓
Hence, f is a many to one function. ✓
Therefore, the inverse of f is not a function.

(b) Find the largest possible domain consisting of both positive and negative numbers so that the inverse of f is a function.

$(-\infty, 1]$ ✓

(c) For the domain you specified in part (b), find the rule for the inverse of f.

Let $y = e^{(x-1)^2}$
$(x-1)^2 = ln\, y$ ✓
$x = 1 \pm \sqrt{ln\, y}$ ✓
But $x \leq 1$, $\Rightarrow x = 1 - \sqrt{ln\, y}$. ✓
Hence, $f^{-1}(x) = 1 - \sqrt{ln\, x}$

---

19. [5 marks: 2, 3]

Consider $f(x) = \left| \frac{2x-1}{x-3} \right|$.

(a) Explain clearly why within its natural domain f does not have an inverse function.

$f(-2) = f(\frac{4}{3}) = 1$ ✓
Hence, f is not a one-to-one function. ✓
Therefore, f does not have an inverse function.

Calculator Free

19. (b) For $f(x) = \left|\dfrac{2x-1}{x-3}\right|$ where $\frac{1}{2} \le x < 3$, find the rule for f^{-1}.

For $\frac{1}{2} \le x < 3$: $y = f(x) = -\left(\dfrac{2x-1}{x-3}\right)$ ✓

$xy - 3y = 1 - 2x$ ✓

$x = \dfrac{1+3y}{2+y}$

Hence, $f^{-1}(x) = \dfrac{1+3x}{2+x}$ ✓

---

20. [6 marks: 3, 3]

Consider $f(x) = \sin 2x$ and $g(x) = \cos \dfrac{x}{2}$.

(a) $f(x)$ is a one-to-one function within the domain $-a \le x \le a$.
Determine the largest possible value for $|a|$.
Hence, determine the rule for $f^{-1}(x)$ and state the corresponding range.

Max value for $|a| = \frac{\pi}{4}$ ✓

$y = \sin 2x \Rightarrow x = \frac{1}{2}\sin^{-1} y$

Hence, $f^{-1}(x) = \frac{1}{2}\sin^{-1} x$. ✓

Range: $[-\frac{\pi}{4}, \frac{\pi}{4}]$ ✓

(b) $g(x)$ is a one-to-one function within the domain $0 \le x \le b$.
Determine the largest possible value for b.
Hence, determine the rule for $g^{-1}(x)$ and state the corresponding range.

Max value for $b = 2\pi$ ✓

$y = \cos \frac{x}{2} \Rightarrow x = 2\cos^{-1} y$

Hence, $f^{-1}(x) = 2\cos^{-1} x$. ✓

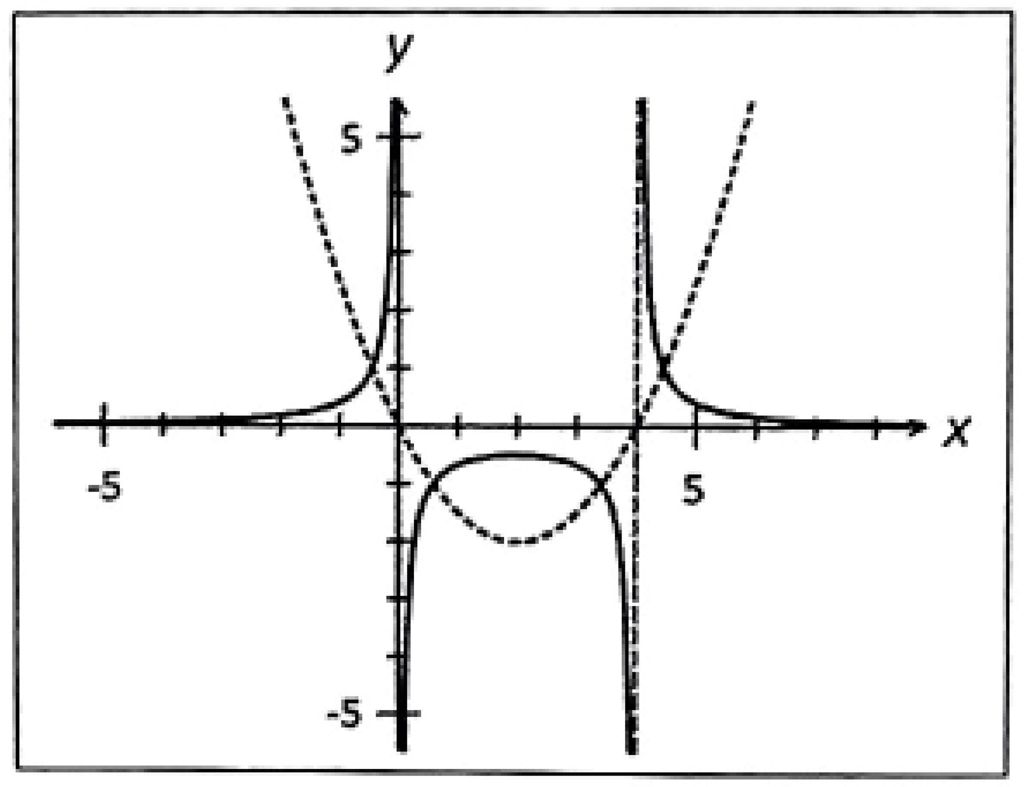
Range: $[0, 2\pi]$ ✓

06 Functions II

Calculator Free

1. [12 marks: 4, 4, 4]

(a) The sketch of $y = f(x)$ is given in the accompanying diagram. Sketch on the same axes the graph of $y = \dfrac{1}{f(x)}$.

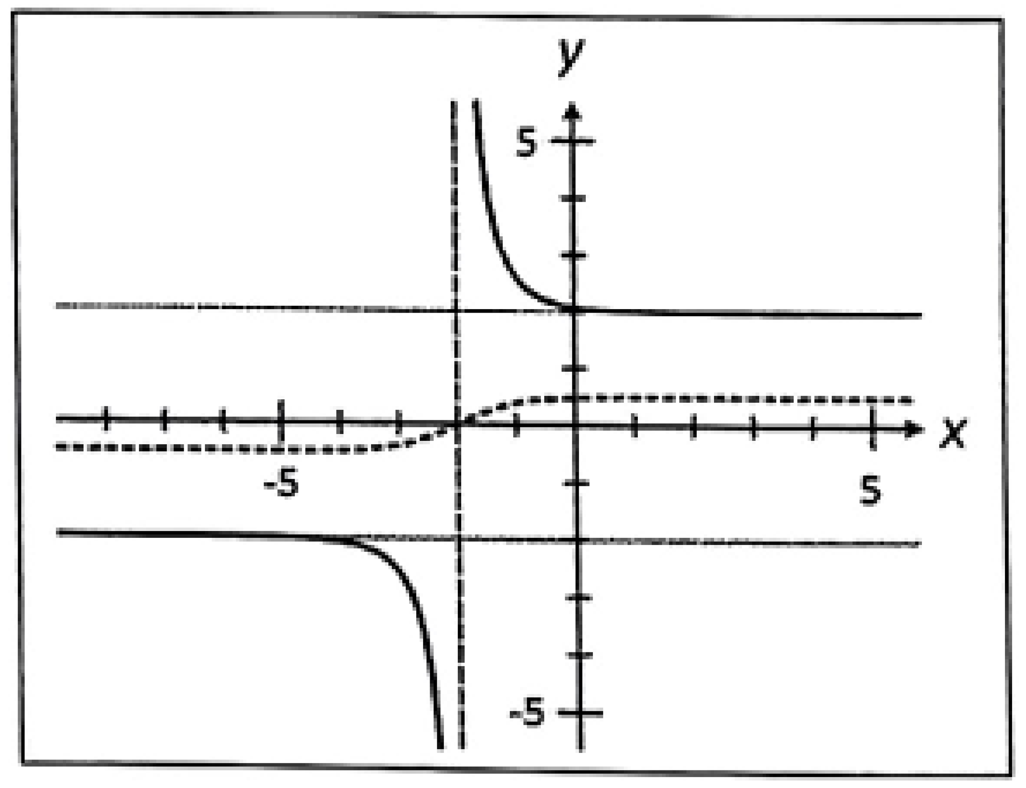


Asymptotes: $y = 0, x = 0, x = 4$ ✓✓

Max point $(2, \frac{1}{2})$ ✓

All correct ✓

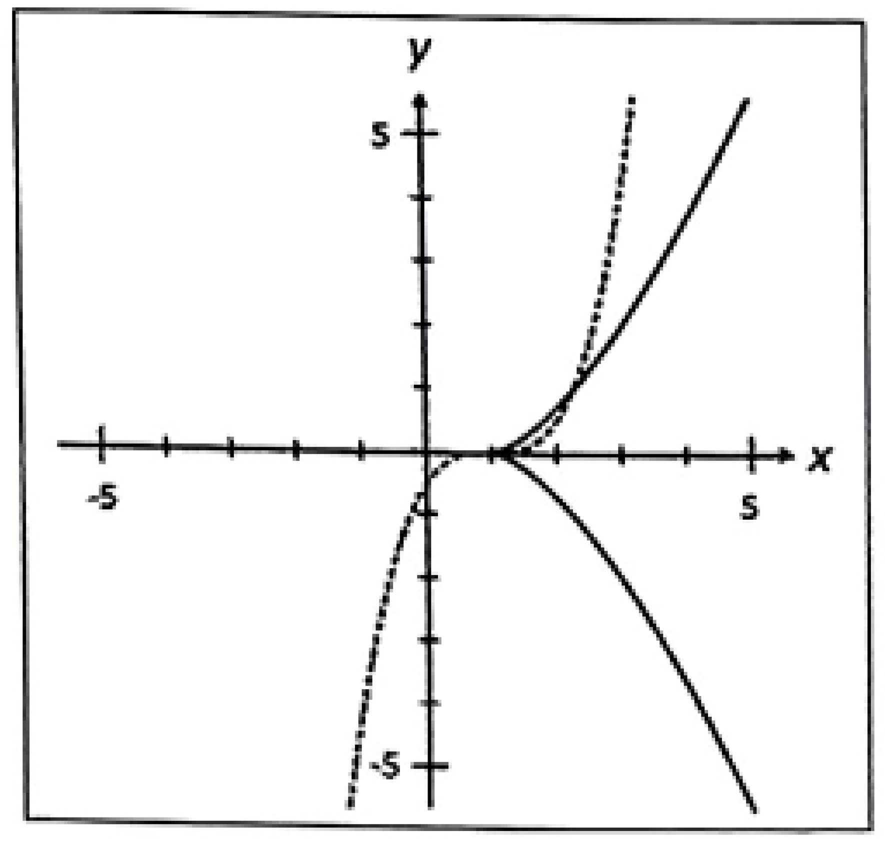
(b) The sketch of $y = \dfrac{1}{f(x)}$ is given in the accompanying diagram. Sketch on the same axes the graph of $y = f(x)$.



Asymptotes: $y = -2, y = 2, x = -2$ ✓✓✓

All correct ✓

(c) The sketch of $y = f(x)$ is given in the accompanying diagram. Sketch on the same axes the graph of $y^2 = f(x)$.



Domain: $[1, \infty)$ ✓

Symmetrical about the x-axis. ✓

Both graphs intersect when $y = 1$ ✓

All correct. ✓